Kleine AT III syllabus - Embedding Calculus

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The theme of this *Kleine Algebraische Topologie* is embedding calculus. Embedding calculus is a means of understanding embedding spaces of manifolds through a certain tower:

$$\begin{tikzcd}
 & \cdots \arrow[d] \\
 & \mathrm{T}_3 \operatorname{Emb}(M,N) \arrow[d] \\
 & \mathrm{T}_2 \operatorname{Emb}(M,N) \arrow[d] \\
\operatorname{Emb}(M,N) \arrow[r] \arrow[ur] \arrow[uur] & \mathrm{T}_1 \operatorname{Emb}(M,N)
\end{tikzcd}$$

Here M and N are smooth manifolds of dimension m and n, respectively, and $\operatorname{Emb}(M,N)$ is the space of embeddings of M into N ($m \leq n$). We will build this tower and find that its layers can in principle be calculated. We will then learn about the Goodwillie-Klein-Weiss theorem, which provides a criterion for the map from $\operatorname{Emb}(M,N)$ to the limit of the tower to be an equivalence. We conclude with a sampling of applications of this framework in computing or better understanding the original space $\operatorname{Emb}(M,N)$.

We will assume a working knowledge of the language of higher category theory.

Talk 1. Foundations, I

The goal of this talk is to leisurely motivate and set up the embedding calculus tower. Start by motivating the space of embeddings; for instance, explain how one can understand diffeomorphism groups of a manifold by understanding embeddings into the manifold by means of the isotopy extension theorem (see [Kup24, Sec. 1.1]). Give examples of embedding spaces ([Kup24, Sec. 1.1]). Motivate embedding calculus as a "pointillistic" viewpoint to understand the space of embeddings ([Kup25, Sec. 1.1]). Carefully set up the ∞-category of d-dimensional manifolds Mfd_d and explain the embedding calculus tower culminating with a description of the first layer (follow ([Kup24, Sec. 1.2.1, 1.2.4]). In particular, explain that equivalences in Mfd_d are isotopy equivalences, and hence that embedding spaces are isotopy invariant. Also remark that the construction can be carried out for manifolds with boundary. However, do not specialise [Kup24, Sec. 2.2.3], but rather the equivalent approach of Boavida de Brito and Weiss, as the latter will be more convenient for applications. The layers of the tower are described through the variant of disc presheaves as specified by the right-hand side of [KK26, (107), Sec. 5.3.3] (take Q to be the empty set).

Talk 2. Foundations, II

Begin by briefly comparing the embedding calculus formalism of the previous talk with that constructed in [BW13] (see [Kup24, Prob. 5, 6]). State the universal properties of the layers of the tower and its limit ([Kup24, Sec. 1.2.2]). We next build up to a description of the higher layers. State [KK26, Thm. 3.8], and then [KK26, Thm. 4.9] specialising the latter to the case of disc presheaves. Sketch how the latter follows from the former. Explain smoothing theory as a consequence of this description ([KK26, Sec. 4.5], [Kup24, Sec. 3.2]), again specialised to disc presheaves. In particular, explain that the smooth embedding calculus tower is base-changed from the continuous one.

Talk 3. Convergence of the Tower

Explain the explicit description of the layers of the embedding tower in the smooth and topological case. A reference is [Kup24, Sec. 3.1, Prob. 13, 14], but more details can be found in [KK26, Lem. 4.11, Prop. 5.12, Ex. 5.14]. With this established, state the main convergence theorem of Goodwillie-Klein-Weiss, and its strengthening due to Krannich-Kupers [Kup24, Sec. 1.2.3] (if time permits, explain the original Goodwillie-Klein multiple disjunction result — for instance, see [KK24, Thm. 2.3] — and how it reduces to the stated version, as in [Kup24, Prob. 7, 8]). As a first application, describe how embedding calculus detects certain exotic spheres by stating and proving [KK24, Thm. C].

Talk 4. Applications

The goal of this talk is to explain some applications of the convergence of embedding calculus. For instance, the talk could comprise the following examples:

1. **Application to knots and links in 4-manifolds:** State and prove [KK24, Thm. A]. The proof has three inputs: the convergence result established in the previous talk, the fact that smooth embedding calculus depends only on the underlying formally smooth manifolds, and an algebraic lifting criterion for homeomorphisms to isomorphisms in the formally smooth category. The second of these follows immediately from smoothing theory as described in Talk 2, so there is no need to go through the arguments of [KK24, Sec. 3].
2. **Contractibility of $\mathrm{Homeo}_\partial(M)$ for high-dimensional contractible M:** State [GR24, Thm. A] and its improvement [KK26, Thm. 6.18]. Sketch the proof, possibly following the outline below:
   - Let M^o denote the manifold obtained by removing a disc from the interior of M. Use the isotropy extension fibre sequence and the Alexander trick to show that $\mathrm{Homeo}_\partial(M) \cong \mathrm{Emb}_\partial(M^o, M)$.
   - By the convergence theorem stated in the previous talk, it follows that $\mathrm{Emb}_\partial(M^o, M)$ can be computed via embedding calculus. Explain that $\partial \to M^o$ is acyclic, and show inductively that every layer in the tower is contractible, following [GR24, Lem. 2.2, Lem. 2.3].
3. **Finiteness of embedding spaces**: State [BKK24, Prop. 4.5], which uses embedding calculus to establish the finiteness of homotopy groups of certain embedding spaces, and sketch its proof. For the latter, you may explain a weaker version of the statement by neglecting fundamental group concerns.

Talk 5: Expert talk

This last talk will be given by Michael Weiss. He will speak about diffeomorphisms of discs in relation to manifold calculus.

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References

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