THE INAUGURAL KLEINE AT WORKSHOP – FALL 2025 EVENNESS VIA SPECTRAL SEQUENCES

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1. Introduction

Spectral sequences are ubiquitous tools for computation in algebra, algebraic geometry, and algebraic topology, providing a systematic way to compute (co)homology by successive approximations. A typical situation in which one encounters a spectral sequence is in the spectral sequence associated to a filtered object of an Abelian category – or, in more modern language, filtered objects in stable ∞-categories with a fixed t-structure.

Let $\mathcal{C}$ be a stable ∞-category with sequential limits and colimits, and fix a t-structure on $\mathcal{C}$. For a filtered object $\mathrm{F}^\star \in \mathrm{Fun}(\mathbb{Z}^{\mathsf{op}}, \mathcal{C})$, the associated graded complex

$$\cdots \to \mathrm{gr}^{-1}[-1] \to \mathrm{gr}^0 \to \mathrm{gr}^1[1] \to \dots$$

gives rise to the E^1-page of the spectral sequence associated to the filtered object $\mathrm{F}^\star$ by passing to homotopy objects (cf. [10, Tag 012K]). The décalage construction exposited by Antieau [2] following work of Deligne [4] and Levine [6] (vis. [1] for an overview) recursively produces the higher pages of the spectral sequence associated to $\mathrm{F}^\star$ along with the corresponding differentials. Moreover, the ∞-categorical enhancements reveal that the décalage functor is in fact lax symmetric monoidal if $\mathcal{C}$ is symmetric monoidal, hence capturing multiplicative phenomena inherent in the spectral sequence.

Building on this framework for spectral sequences arising from filtered objects, one can investigate specific filtrations of interest in (∞-categorical) algebra. The even filtration of Hahn-Raksit-Wilson [5] generalizes the so-called "motivic" filtration on topological Hochschild homology developed by Bhatt-Morrow-Scholze [3] which has led to a number of recent developments in p-adic geometry. Defining the even filtration of an $\mathbb{E}_\infty$-ring A to be the limit over all maps of $\mathbb{E}_\infty$ rings $A \to B$ of the double-speed Postnikov filtration

$$\mathrm{Fil}^\star_{\mathrm{ev}}(A) = \lim_{A \to B, B \text{ even}} \tau_{\geq 2\star}(B),$$

the authors prove that several previously known constructions can be unified under the even filtration and computed with its associated spectral sequence.

The inaugural Kleine AT workshop will study the aforementioned works of Antieau and Hahn-Raksit-Wilson, providing a working knowledge of spectral sequence computations by showcasing a topic of active interest in homotopy theory.

Prerequisites. Participants are expected to come with a strong working knowledge of ordinary category theory and homological algebra, as well as familiarity with basic constructions in higher category theory and higher algebra as in [7, 8].

2. Schedule of Talks

References are to [2] for talks 1-3 and to [5] for talks 4-5, unless otherwise stated.

2.1. Talk 1: The Filtered Derived Category and the Beilinson t-Structure.

Introduce the filtered derived category from [3.1-3.9] or from [9]. Discuss the algebraic properties of graded and filtered objects such as associated gradeds, coherent cochain complexes, and its connection to complete decreasing filtrations [3.16-3.22]. Construct the Beilinson t-structure and give an overview of related constructions [3.24-3.30].

2.2. Talk 2: Décalage I.

Introduce the definition of a spectral sequence and the construction of a spectral sequence from filtered objects [4.1-4.5]. Define the décalage functor as the realization of the Whitehead tower of the filtered object and the décalage construction of the spectral sequence [4.5-4.9]. Recall Lurie's definition of a spectral sequence [4.10-4.11].

2.3. Talk 3: Décalage II.

State the comparison theorem between the décalage spectral sequence and the spectral sequence of a filtered object [4.13] and give the pictorial proof as in [4.25] and [Figure 4], recalling the preliminaries [4.15-4.24] as needed. Compare the décalage as defined using the Beilinson t-structure and as defined by Deligne [5.1-5.5].

2.4. Talk 4: Evenness I.

Give some basic intuition for $\mathbb{E}_\infty$ rings as spectra where $\pi_0(R)$ is a commutative ring and $\pi_n(R)$ a $\pi_0(R)$-module for each n with graded-commutative multiplication [7, Rmk 7.1.1.6]. Introduce the notion of evenness of $\mathbb{E}_\infty$-rings and discuss how they arise naturally in algebraic topology (eg. MU) [1.1.1-1.1.2]. State and prove results relating to the commutative algebra of even rings [2.1.1-2.1.6, 2.1.8]. Conclude by constructing the faithfully flat topology [2.2.4, 2.2.6, 2.2.9-2.2.11] (ie. ignore the p-complete and S^1 settings).

2.5. Talk 5: Evenness II.

Define even faithfully flat maps and evenly free maps [2.2.15] and the connection between even freenees and even faithful flatness [2.2.16] with even faithfully flat descent [2.2.17]. Briefly introduce the Adams spectral sequence as a black box. Show that the unit map $\mathbb{S} \to \mathrm{MU}$ is evenly free [2.2.20] and prove Novikov descent [2.2.21]. Recover the even filtration of the sphere spectrum as the décalage of the Adams-Novikov filtration [1.1.5].

References

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